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Quantum Thermal Effects of a Radiating Rotating Charged Black Hole

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Asymptotic solutions of the Klein–Gordon equation in a region near the event horizon of a radiating rotating charged black hole are obtained by using generalized tortoise coordinates. The location of the event horizon and the Hawking temperature of the black hole are given. Both the horizon and the temperature depend on the angle and time, due to radiation. However, they are independent of the angle if either rotation or radiation vanishes. The treatment encompasses as special cases the results on a number of well-known black holes.

1. INTRODUCTION

Since Hawking's (1975) original discovery of black hole thermal radiation by using techniques of quantum field theory on a given classical background, quantum thermal radiation due to a black hole has been studied by different authors in different types of spacetime, such as the Kerr (Zhao and Guai, 1983; Liu and Xu, 1980), Kerr–Newman (Damour and Ruffini, 1976; Zhao *et al.*, 1981), NUT–Kerr–Newman (Ahmed, 1987), Kerr–Newman–Kasuya (Ahmed and Mondal, 1993), Vaidya–Schwarzschild–de Sitter (Dai *et al.*, 1993), and Vaidya–Bonner (Dai and Zhao, 1992) spacetimes. In the present paper we use a new method proposed by Zhao, Dai, and Yang (Zhao and Dai, 1991, 1992) to study the Hawking radiation of a radiating rotating charged black hole. The metric which describes the external gravitational field of such a black hole has been obtained by us recently (Jing and Wang, 1996). The investigation of quantum effects in a radiating rotating charged black hole is interesting for the following reasons: (a) As is well known, black holes have, in general, radiation due to dynamical evolution of Hawking

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evaporation (Birrell and Davies, 1982; William and Weems, 1990). Hence, the gravitational field surrounding a rotating charged black hole cannot be described by the stationary Kerr–Newman metric except in the approximation in which one neglects the energy density of the emitted radiation. Such an approximation may not be valid for certain processes, in which case the radiation must be taken into account. Fortunately, the radiating rotating charged metric can be used to model the dynamical evolution of the evaporating charged black hole. (b) We hope that some new results may be obtained due to the radiation. (c) The results of the radiating rotating charged black hole will include the results of a number of well-known black holes as special cases.

The plan of the paper is as follows. Section 2 deals with the structure of a quite general spacetime of a radiating rotating charged black hole. In Section 3 the Klein–Gordon equation in the region near the event horizon of the black hole is solved, and the location of the event horizon and the Hawking temperature are given; a brief discussion of the results obtained is given.

2. METRIC OF A RADIATING ROTATING CHARGED BLACK HOLE

The physical line element describing a radiating rotating charged black hole is given in the form (Jing and Wang, 1996)

$$g_{\mu\nu} = \begin{pmatrix} 1-(2mr-Q^2)\rho\bar{\rho} & -1 & 0 & -a(2mr-Q^2)\rho\bar{\rho}\sin^2\theta \\ -1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & -\dfrac{1}{\rho\bar{\rho}} & 0 \\ -a(2mr-Q^2)\rho\bar{\rho}\sin^2\theta & -a\sin^2\theta & 0 & \sin^4\theta\left[(Q^2-2mr)a^2\rho\bar{\rho}-\dfrac{a^2+r^2}{\sin^2\theta}\right] \end{pmatrix} \tag{2.1}$$

It is a natural nonstationary generalization of the Kerr–Newman metric, where the latter has the same form but with constant m and Q. In metric (2.1) the parameter v is the Eddington–Finkelstein-type advanced time, and $m(v)$ and $Q(v)$ are the mass and charge of the radiating rotating charged body as seen by an observer at infinity, respectively; they are arbitrary functions of the advanced time coordinate. The total angular momentum of the body $J(v)$ is given by $m(v)a$, where a is a constant just as in the Kerr–Newman case.

The contravariant components of the metric (2.1) are given by

$$g^{\mu\nu} = \begin{pmatrix} -a^2 \sin^2\theta\, \rho\bar{\rho} & -(a^2 + r^2)\rho\bar{\rho} & 0 & a\rho\bar{\rho} \\ -(a^2 + r^2)\rho\bar{\rho} & (2mr - r^2 - a^2 - Q^2)\rho\bar{\rho} & 0 & a\rho\bar{\rho} \\ 0 & 0 & \rho\bar{\rho} & 0 \\ a\rho\bar{\rho} & a\rho\bar{\rho} & 0 & -\dfrac{\rho\bar{\rho}}{\sin^2\theta} \end{pmatrix} \tag{2.2}$$

The classical properties of the solution are studied in detail by Jing and Wang (1992). We can use the metric (2.1) to model the dynamical evolution of the evaporating rotating charged black hole. The spacetime geometry of the black hole is characterized by two surfaces of particular interest: the apparent horizon r_A and the event horizon r_h. These coincide at $r = 2M$ for a classical Schwarzschild hole. In general, these surfaces, all of which will be regarded as three-dimensional histories of topological spherical two-surfaces, do not coincide. In the spacetime (2.1) the apparent horizon r_A is a timelike surface which can be found from $g_{vv} = 0$. The event horizon is necessarily a null surface and is defined by the outermost locus traced by outgoing photons that can "never" reach arbitrarily large distance. From the equation of the null hypersurface condition

$$g^{\mu\nu} \frac{\partial f}{\partial x^\mu} \frac{\partial f}{\partial x^\nu} = 0$$

and metric (2.2) we obtain the equation determining the event horizon as follows:

$$a^2 \sin^2\theta \dot{r}_h^2 - 2(r_h^2 + a^2)\dot{r}_h + (r_h^2 + a^2 + Q^2 - 2mr_h) + r_h'^2 = 0 \tag{2.3}$$

Here and hereafter

$$\dot{r}_h = \left(\frac{dr}{dv}\right)_{r=r_h} \quad \text{and} \quad r_h' = \left(\frac{dr}{d\theta}\right)_{r=r_h}$$

Equation (2.3) shows that the shape of the event horizon of the black hole is not spherically symmetric and depends on the time and the angle, due to the radiation.

3. QUANTUM THERMAL RADIATION OF THE RADIATING ROTATING CHARGED BLACK HOLE

The Klein–Gordon equation is expressed as

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \Phi}{\partial x^{\nu}} \right) - \mu^2 \Phi = 0 \tag{3.1}$$

where μ represents the mass of the Klein–Gordon particles. Substituting the metric (2.1) into (3.1), we have

$$\begin{aligned} &\left\{ -a^2 \sin^2\theta \frac{\partial^2 \Phi}{\partial v^2} + (2mr - Q^2 - r^2 - a^2) \frac{\partial^2 \Phi}{\partial r^2} - \frac{\partial^2 \Phi}{\partial \theta^2} - \frac{1}{\sin^2\theta} \frac{\partial^2 \Phi}{\partial \varphi^2} \right. \\ &\left. - 2(r^2 + a^2) \frac{\partial^2 \Phi}{\partial v \, \partial r} + 2a \frac{\partial^2 \Phi}{\partial v \, \partial \varphi} + 2a \frac{\partial^2 \Phi}{\partial r \, \partial \varphi} \right\} \sin\theta - 2r \sin\theta \frac{\partial \Phi}{\partial v} \\ &+ 2(m - r) \sin\theta \frac{\partial \Phi}{\partial r} + \cos\theta \frac{\partial \Phi}{\partial \theta} - \left(\mu^2 \frac{\sin\theta}{\rho\bar{\rho}} \right) \Phi = 0 \end{aligned} \tag{3.2}$$

In order to solve equation (3.2) in the region near the event horizon, we introduce the generalized tortoise coordinate transform (Zhao and Guei, 1983; Liu and Xu, 1980; Dai *et al.*, 1993)

$$\begin{aligned} r_* &= r + \frac{1}{2\kappa} \ln[r - r_h(v, \theta)] \\ v_* &= v - v_0, \qquad \theta_* = \theta - \theta_0 \end{aligned} \tag{3.3}$$

where r_h represents the location of the event horizon which is determined by equation (2.3), and the parameter κ is unchanged under the tortoise coordinate transform. We will find that κ is a temperature function. Using the coordinate transform (3.3), we can cast equation (3.2) into the form

$$\begin{aligned} &\frac{1}{F} \left(-a^2 \sin^3\theta \frac{\partial^2 \Phi}{\partial {v_*}^2} + \sin\theta \left\{ (2mr - Q^2 - r^2 - a^2) \left[1 + \frac{1}{2\kappa(r - r_h)} \right]^2 \right. \right. \\ &- \left(\frac{r_k'}{2\kappa(r - r_h)} \right)^2 - a^2 \sin^2\theta \left(\frac{\dot{r}_h}{2\kappa(r - r_h)} \right)^2 + (a^2 + r^2) \frac{\dot{r}_h}{\kappa(r - r_h)} \\ &\left. \times \left[1 + \frac{1}{2\kappa(r - r_h)} \right] \right\} \frac{\partial^2 \Phi}{\partial r_*^2} - \sin\theta \frac{\partial^2 \Phi}{\partial \theta_*^2} - \frac{1}{\sin\theta} \frac{\partial^2 \Phi}{\partial \varphi_*^2} + \frac{2 \partial^2 \Phi}{\partial r_* \, \partial v_*} \\ &+ \frac{2a \sin\theta \, \partial^2 \Phi}{\partial v_* \, \partial \varphi} + \frac{\sin\theta r_h'}{\kappa(r - r_h)} \frac{\partial^2 \Phi}{\partial r_* \, \partial \theta_*} + \frac{a \sin\theta \{ 1 + 2\kappa(r - r_h) - \dot{r}_h \}}{\kappa(r - r_h)} \end{aligned}$$

$$\times \frac{\partial^2\Phi}{\partial r_* \, \partial\varphi} + \left\{ a^2 \sin^3\theta \left[\frac{\ddot{r}_h(r - r_h) + \dot{r}_h^2}{2\kappa(r - r_h)^2} \right] - \frac{(2mr - Q^2 - r^2 - a^2) \sin\theta}{2\kappa(r - r_h)^2} \right.$$

$$+ \sin\theta \, \frac{r_h''(r - r_h) + r_h'^2}{2\kappa(r - r_h)^2} - \frac{\sin\theta \, (a^2 + r^2)\dot{r}_h}{\kappa(r - r_h)^2} + \frac{2r \sin\theta \, \dot{r}_h}{2\kappa(r - r_h)} + 2(m - r) \sin\theta$$

$$\times \left[1 + \frac{1}{2\kappa(r - r_h)} \right] - \left. \frac{\cos\theta \, r_h'}{2\kappa(r - r_h)} \right\} \frac{\partial\Phi}{\partial r_*} - 2r \sin\theta \, \frac{\partial\Phi}{\partial v_*} - \cos\theta \, \frac{\partial\Phi}{\partial\theta_*}$$

$$\left. - \frac{\mu^2 \sin\theta}{\rho\bar{\rho}} \Phi \right) = 0 \tag{3.4}$$

with

$$F = \sin\theta \left\{ -(r^2 + a^2) \left[1 + \frac{1}{2\kappa(r - r_h)} \right] + a^2 \sin^2\theta \, \frac{\dot{r}_h}{2\kappa(r - r_h)} \right\}$$

Making use of the equation of the event horizon (2.3), we know that the coefficient of the term $\partial^2\Phi/\partial r_*^2$

$$\left\{ -(r^2 + a^2) \left[1 + \frac{1}{2\kappa(r - r_h)} \right] + a^2 \sin^2\theta \, \frac{\dot{r}_h}{2\kappa(r - r_h)} \right\}^{-1}$$

$$\times \left\{ (2mr - Q^2 - r^2 - a^2) \left[1 + \frac{1}{2\kappa(r - r_h)} \right]^2 - a^2 \sin^2\theta \left(\frac{\dot{r}_h}{2\kappa(r - r_h)} \right)^2 \right.$$

$$\left. - \left(\frac{r_h'}{2\kappa(r - r_h)} \right)^2 + (a^2 + r^2) \frac{\dot{r}_h}{\kappa(r - r_h)} \left[1 + \frac{1}{2\kappa(r - r_h)} \right] \right\} \tag{3.5}$$

is an indefinite form as $r \rightarrow r_h(v_0, \theta_0)$. Using the L'Hôspital rule and adjusting the value of the limit as $r \rightarrow r_h(v_0, \theta_0)$ (Damour and Ruffini, 1976; Zhao *et al.*, 1981; Dai *et al.*, 1993), we obtain

$$\kappa = \left\{ \frac{(m - r_h) - 2r_h\dot{r}_h}{[a^2 \sin^2\theta - (a^2 + r_h^2)]\dot{r}_h - (4mr_h - 2Q^2 - a^2 - r_h^2)} \right\}_{v=v_0, \theta=\theta_0} \tag{3.6}$$

It is easy to show that, as $r \rightarrow r_h(v_0, \theta_0)$, the coefficients of the terms $\partial^2\Phi/\partial r_* \, \partial\theta_*$, $\partial^2\Phi/\partial r_* \, \partial\varphi_*$, and $\partial\Phi/\partial r_*$ can respectively be expressed as

$$\alpha = \frac{2r_h'}{a^2 \sin^2\theta_0 \, \dot{r}_h - (a^2 + r_h^2)}$$

$$\beta = \frac{2a(1 - \dot{r}_h)}{a^2 \sin^2\theta \, \dot{r}_h - (a^2 + r_h^2)}$$

$$\gamma = \frac{a^2 \sin^2\theta_0 \, \ddot{r}_h + r_h'' - 6r_h\dot{r}_h - \operatorname{cotg}\theta_0 \, r_h'}{a^2 \sin^2\theta_0 \, \dot{r}_h - (a^2 + r_h^2)} \tag{3.7}$$

and the coefficients of the terms $\partial^2\Phi/\partial v_*^2$, $\partial^2\Phi/\partial\theta_*^2$, $\partial^2\Phi/\partial\varphi^2$, $\partial\Phi/\partial v_*$, $\partial\Phi/\partial\theta_*$, and Φ tend to zero. Thus, as $r \to r_h(v_0, \theta_0)$, equation (3.4) reduces to the following standard form of the wave equation:

$$\frac{\partial^2\Phi}{\partial r_*^2} + 2\frac{\partial^2\Phi}{\partial r_*\,\partial v_*} + \alpha\frac{\partial^2\Phi}{\partial r_*\,\partial\theta_*} + \beta\frac{\partial^2\Phi}{\partial r_*\,\partial\varphi} + \gamma\frac{\partial\Phi}{\partial r_*} = 0 \qquad (3.8)$$

After separating the variables

$$\Phi = R(r_*)\Theta(\theta_*)\Psi(\varphi)\exp(-i\omega v_*) \qquad (3.9)$$

we find that (3.8) reads

$$\left(\frac{1}{\partial R/\partial r_*}\right)\frac{\partial^2 R}{\partial r_*^2} + \gamma - 2i\omega = -\left(\alpha\frac{\partial\Theta}{\Theta\,\partial\theta_*} + \beta\frac{\partial\Psi}{\Psi\,\partial\varphi}\right) \qquad (3.10)$$

This shows that both sides are equal to the same complex number $\lambda + i2\sigma$. Then equation (3.10) can be rewritten as

$$\frac{\partial^2 R}{\partial r_*^2} + [\lambda + \gamma - 2i(\omega - \sigma)]\frac{\partial R}{\partial r_*} = 0$$

$$\alpha\frac{\partial\Theta}{\Theta\,\partial\theta_*} + \beta\frac{\partial\Psi}{\Psi\,\partial\varphi} = \lambda + 2i\sigma \qquad (3.11)$$

Two linearly independent radial solutions of (3.11) are given by

$$\psi_\varphi^{\text{in}} \sim e^{-i\omega v}$$

$$\psi_\omega^{\text{out}} \sim e^{-i\omega v}e^{-(\lambda+y)r}e^{2i[\omega-\sigma]r} \qquad (3.12)$$

where ψ_ω^{in} represents an incoming wave and is analytic on the event horizon; $\omega_\omega^{\text{out}}$, however, represents an outgoing wave and has a logarithmic singularity on the horizon. Just outside the event horizon, substituting (3.3) into (3.12) and noting $r_* \sim (1/2\kappa)\ln(r - r_h)$, on the event horizon we have

$$\psi_\omega^{\text{out}} \sim \exp(-i\omega v_*)(r - r_h)^{-(\lambda+\gamma)/(2\kappa)}(r - r_h)^{i(\omega-\sigma)/\kappa} \qquad (3.13)$$

We now extend the outgoing wave outside the horizon to the region inside. Since on the event horizon the outgoing wave function is not analytic and cannot be extended straightforwardly to the region inside, it must be continued analytically in the complex plane by going around the event horizon. Hence inside the event horizon

$$\tilde{\psi}_\omega^{\text{out}} \sim \exp(-i\omega v_*)(r_h - r)^{-(\lambda+\gamma)/(2\kappa)}(r_h - r)^{i(\omega-\sigma)/\kappa}\exp\left(\frac{\pi(\omega - \sigma)}{\kappa}\right) \qquad (3.14)$$

Introducing the step function

$$y(x) = \begin{cases} 1 & \text{for} \quad x \geq 0 \\ 0 & \text{for} \quad x < 0 \end{cases} \tag{3.15}$$

we can generally express the outgoing wave function as

$$\Psi_\omega^{\text{out}} = N\left\{y(r - r_h)\psi_\omega^{\text{out}}(r - r_h) + y(r_h - r)\psi_\omega^{\text{out}}(r_h - r) \exp\left[\frac{\pi(\omega - \sigma)}{\kappa}\right]\right\} \tag{3.16}$$

where Ψ_ω^{out} is the normalized Klein–Gordon wave function. According to the suggestion of Damour and Ruffini (1976) and Sannan (1988), and using the normalization condition, we obtain

$$N^2 = \frac{\Gamma_\omega}{\exp(2\pi(\omega - \sigma)/\kappa) - 1} = \frac{\Gamma_\omega}{\exp((\omega - \sigma)/k_B T) - 1} \tag{3.17}$$

with

$$T = \frac{\kappa}{2\pi k_B} \tag{3.18}$$

where Γ_ω is the transmission coefficient caused by the potential barrier in the exterior gravitational field; k_B is the Boltzmann constant. Equation (3.17) is the main formula demonstrating the emission of a thermal spectrum of Klein–Gordon particles of the radiating rotating charged black hole. The temperature of the thermal radiation is shown by (3.18) and the parameter κ introduced in (3.3) is a temperature function. Equation (3.6) shows that κ depends on the time and the angle, due to the radiation. It is interesting that κ is independent of the angle if either rotation or radiation vanishes. For a stationary extreme Kerr–Newman black hole ($r^h = m$), $\kappa = 0$, but for the black hole, (3.6) shows that $\kappa \neq 0$ when $r_h = m$.

Our work includes the results of:

(a) The stationary Kerr–Newman black hole (Damour and Ruffini, 1976; Zhao *et al.*, 1981) for m = const and Q = const.

(b) The stationary Kerr black hole (Zhao and Guei, 1983; Liu and Xu, 1980) when m = const and $Q = 0$.

(c) The Reissner–Nordström black hole if m = const, Q = const, and $a = 0$.

(d) The Schwarzschild black hole (Zhao and Guei, 1983; Liu and Xu, 1980) provided m = const, $Q = 0$, and $a = 0$.

(e) The nonstationary Kerr black hole if $Q = 0$.

(f) The radiating Vaidya–Bonner black hole (Dai and Zhao, 1992) when $a = 0$.

(g) The radiating Vaidya black hole (Balbinat, 1986) for $a = Q = 0$.

To summarize, the Hawking radiation of the radiating rotating charged black hole was studied by solving the Klein–Gordon equation in the region near the event horizon with the generalized tortoise coordinates. The location of the event horizon and the Hawking temperature of the black hole are given. Both the event horizon and the Hawking temperature depend on the angle and time, due to radiation. However, r_h and κ are independent of the angle when either radiation or rotation vanishes. The treatment encompasses the results of a number of well-known black holes as special cases.

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